Activity 3 - The Lorentz Transformation

Lorentz transformations are the relativistic equivalent of the classical Galilean transformation, where, due to time dilation and length contraction, there are added complicating effects.

We earlier said that we need to transform intervals when comparing different inertial frames. This is true, but we can get around this by considering the interval from the origin of our coordinate system to the event.

Einstein's original derivation of the transformations between inertial frames was based on the transformation having to preserve certain properties. While rigorous, this is very complicated, so we will use a less rigorous but much easier 'derivation'.

There is an inertial frame S, with coordinates $t,\ x,\ y,\ z$.
There is another inertial frame, travelling at v along the x axis of S, with the x,y & z axes of each frame parallel to those of the other, where at $t = t' = 0$ the origins coincide:

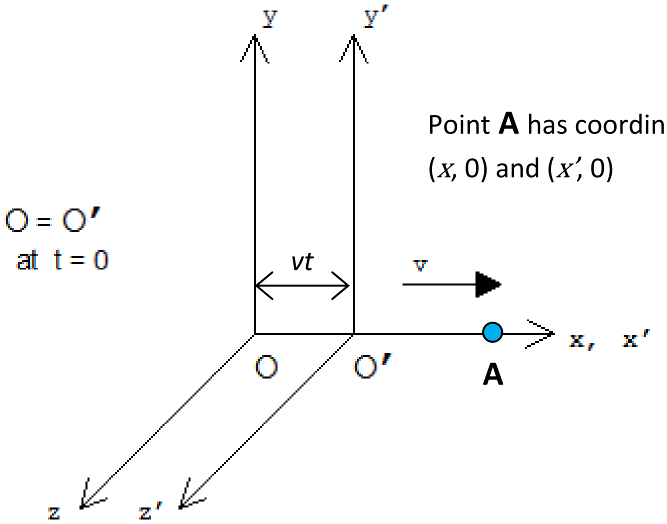


Figure 1. Two Inertial Frames in Standard Configuration

The coordinates in this frame are t', x', y', z'.

We want to be able to find a general equation to relate x' to x, t' to t etc.

We start in the frame S, at time t and at spatial coordinates $(x, 0, 0)$, and make a mark on the x' axis of S'.

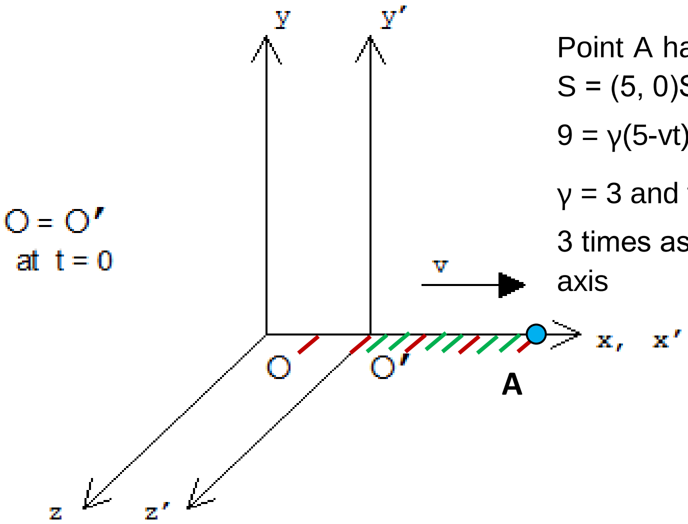
Immediately we know that there will be an $x - vt$ term in our equation, because S' has moved on by amount vt, but this would just be a Galilean transformation!
We know from the previous sections that lengths are contracted in the direction of travel, relative to those in other frames.
Here the observer in x sees the moving x' axis is contracted, so for a given distance x, x' will be bigger than $x - vt$, just like if you squashed a spring and made two marks on it, then let loose: when it extends your marks are further apart.
We have already worked out the factor for length contraction: it is the gamma factor, or as it is sometimes know, the Lorentz factor, γ.
We know that γ is always greater than one, so the equation must be:

$x' = \gamma(x - vt)$.

This is the correct transformation for inertial frames in standard configuration!

You might be worrying that earlier, when we derived length contractions, we had $d = \gamma d'$, and now we have the γ on the other side. This is not a mistake, and is well worth getting your head around.

- We have said that the x' axis contracts. Indeed, in accordance with our length contraction, it becomes shorter by a factor of γ.
- So each of the "ticks" on the x' axis are also moved closer together in this process.



Point A has coordinates
S = (5, 0)S' = (9, 0)
9 = γ(5-vt) = γ(5-2)
γ = 3 and there are indeed
3 times as many marks on our x' axis

- So when we make our mark on the x' axis at point $(x, 0,0)$ in S, we will have more x' ticks the more the axis is contracted.

- Since when we contract the x' axis no ticks are lost, their density must increase by a factor of γ (as seen by observers in S, obviously in S' they are not contracted at all), and so we have γ times the number of ticks we have in x (see diagram).

What if the observer in S' does the exact same thing? All of the logic follows as before, but the direction of motion is reversed, and we have $x' + vt'$,so we obtain:

$$x = \gamma(x' + vt').$$

It is a common feature for all of the Lorentz transformation equations, that to get the inverse transformation you just have to change the sign of v.

To find the time transformations we can just solve the two simultaneous equations for the x and x' transformations:

$x = \gamma(\gamma(x - vt) + vt'),$

$\Rightarrow \; t' = \gamma(t - \frac{vx}{c^2}).$

Make sure you work through the algebra for that yourself. All you need to do is rearrange the first equation, remembering that

$$\gamma \; = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

We can get the inverse transformation by applying our rule of reversing the sign of v:

$$t = \gamma(t' + \frac{vx'}{c^2}).$$

What about y and z directions? Well by symmetry they don't change under a Lorentz transformation: if they did, then if we redefined our x and x' axes by multiplying by -1 (which as the axes are just a grid of numbers we use to locate events in space-time we are allowed to do) we would expect a change in y and y', but we can see that as long as the direction of v is the same then there should be no change in y or y'. This is like having two equations saying:

$$a = b - c$$
$$a = b + c$$

which would clearly imply that $c = 0$.

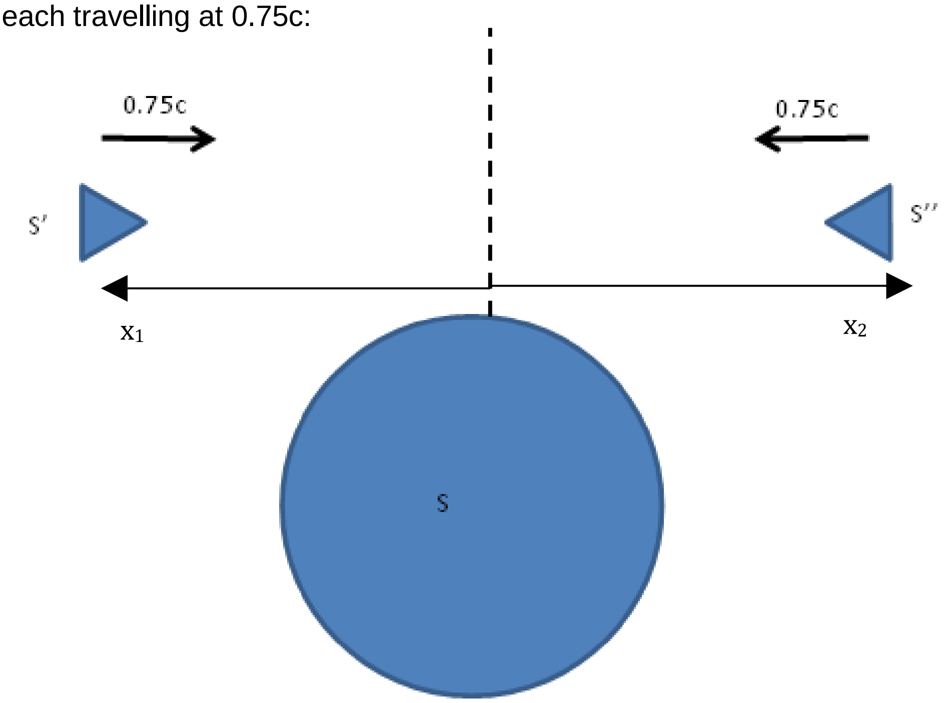
$$\Rightarrow \; y' = y \quad \& \quad z' = z$$

Worked Example - addition of velocities:

If you've been thinking about the concepts above, you may have wondered what happens when two spaceships are travelling towards each other.

In classical physics, that describes our day to day lives, we would just add the speeds: two cars driving towards each other, each at 60mph, are closing at 120mph. Each driver has their own inertial frame of the car, where they can consider themselves to be stationary: the road beneath them is travelling at 60mph, the other car towards them at 120mph.

But what if two spaceships are flying towards each other and from the frame of the Earth they are each travelling at 0.75c:



How fast does the observer in S' see the ship S'' approaching? The answer is **not** 1.5c, because nothing can travel faster than the speed of light in special relativity.

We can write the interval $x_2 - x_1 = \Delta x$, and so we can write speed as:

$$u = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t}$$

since these are inertial frames, their speeds are constant.

Now let us call the speed of S', as seen in frame S, u, and call the speed of S'' as seen in S v.

$$v' = \frac{\Delta x'}{\Delta t'}$$

$$= \frac{\gamma(\Delta x + u\Delta t)}{\gamma\left(\Delta t + \frac{u\Delta x}{c^2}\right)}$$

Where we have used the Lorentz transformations for $\Delta x'$ and $\Delta t'$ noting the direction of v is opposite to u, hence the + instead of -. This is the speed of S'', in the frame S'.

Divide top and bottom by Δt:

$$v' = \frac{v+u}{1+\frac{vu}{c^2}}.$$

Which, substituting in the values for u and v gives $v' = 0.96c$.
This method of writing the speed as change in speed over time only works for constant velocities. It can be extended however to cover accelerating objects, by noting that as long as the acceleration is uniform, $a = \frac{dv}{dt}$. note that we can transform accelerations between two different inertial frames, but both of those frames cannot be accelerating, and we cannot transform into the accelerating frame (without using maths beyond this material). So we could ask how quickly a rocket was accelerating from the Earth frame compared to the frame of a space station flying away from the Earth at constant speed, but we could not find how quickly the space station was moving from the perspective of the accelerating space ship, as it is not an inertial frame.