Worked Examples

Time Dilation

Imagine we had a spaceship which could instantaneously accelerate to whatever speed we wanted (below the speed of light).

Suppose we also found a star 10 000 light years (lyr) away which we wanted to visit.

We decide that we want our journey to take 10 years (in the frame of the spaceship).

How fast do we need to travel, and how long does it take for us to reach the star in the Earth's frame?

Solution:

- We know that the time interval in the spaceship, t', and on the Earth, t, are related by $t = \gamma t'$.
  (remembering that clocks tick slower on the moving object)
- We also know that in the Earth frame the spaceship will take time $t = \frac{d}{v}$ to reach the star, where $d = 10{,}000\ lyr$ and v is as yet unknown.
- Expand γ, and the two equations can be combined to give:

$$t' = \frac{d}{v}\sqrt{1 - \frac{v^2}{c^2}}.$$

- Square both sides and rearrange:

$$\frac{v^2}{c^2} = \frac{d^2}{d^2 + c^2 \mathrm{t}'^2}$$

- Now we could convert everything in SI units and back again, or we could work in units of years and light years. Clearly $c = 1\ lyr\ yr^{-1}$.Substituting in the values from above:

$$\frac{v}{c} = \sqrt{\frac{(10{,}000\ lyr)^2}{(10{,}000\ lyr)^2\ +\ (1\ lyr\ yr^{-1})^2(10\ yr)^2}} = 0.999{,}999{,}5$$

- Where the answer is expressed as a fraction of the speed of light. Needless to say, this is very fast!
- Still working in years and light years we can now easily find the time in the Earth frame, t:

$$t = \frac{d}{v} = \frac{10{,}000\ lyr}{0.999{,}999{,}5\ lyr\ yr^{-1}} = 10{,}000.005\ yr$$

This is in fact the starting point of the Twin Paradox mentioned in many of the links in this document. Again we emphasise that the 10 years on the spaceship are not just an illusion: people would really only age 10 years while 10,000 years passed on Earth.

While the idea of an instantaneously accelerated spaceship is used here to simplify the maths, the same principle holds for a ship which accelerated until it was halfway to the star, then decelerated the rest of the way. This would allow a much more comfortable and achievable acceleration.

Length contraction

A train is 120m long and travelling at speed v. It is coming up to a station, where it won't stop, which has a platform 100m long. How fast does the train need to be going for it to be the same length as the platform, from the inertial frame of the platform?

We call the platform frame S, and the train frame S'. From our length contraction derivation we know that the observer on the platform sees the train contracted therefore $l' > l$ which gives $\gamma l = l'$,so:

$$\frac{l'}{l} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\frac{v}{c} = \sqrt{1 - \frac{l^2}{l'^2}}$$

So substituting the values:

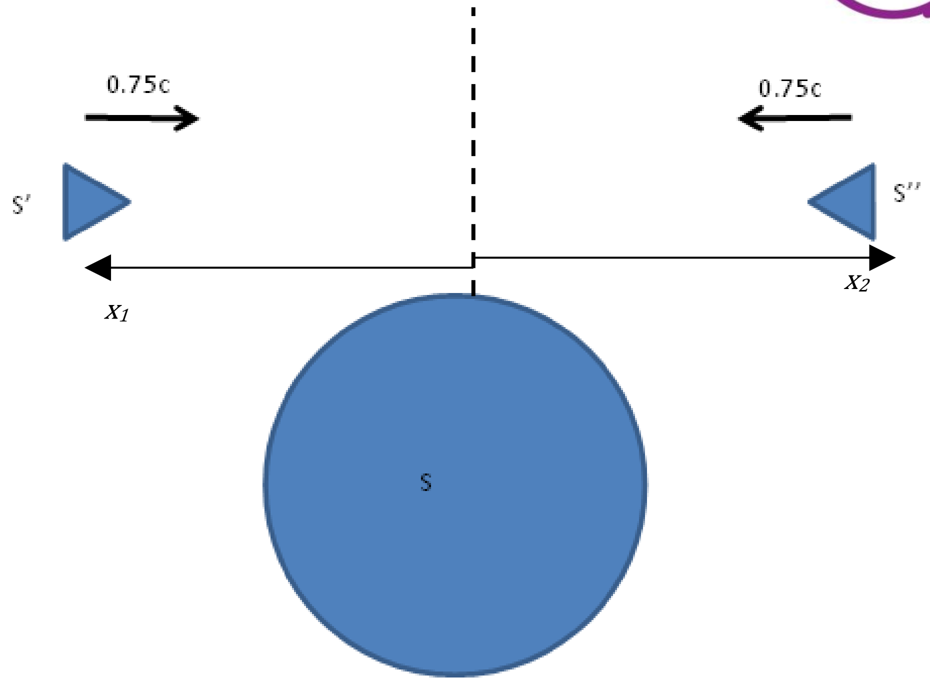
$$\frac{v}{c} = \sqrt{1 - \frac{10{,}000}{14{,}400}} = 0.55c.$$

Addition of velocities

If you've been thinking about the concepts above, you may have wondered what happens when two spaceships are travelling towards each other.

In classical physics, that describes our day to day lives, we would just add the speeds: two cars driving towards each other, each at 60mph, are closing at 120mph. Each driver has their own inertial frame of the car, where they can consider themselves to be stationary: the road beneath them is travelling at 60mph, the other car towards them at 120mph.

But what if two spaceships are flying towards each other, and from the frame of the Earth they are each travelling at 0.75c:



How fast does the observer in S' see the ship S'' approaching? The answer is **not** 1.5c, because nothing can travel faster than the speed of light in special relativity.

We can write the interval $x_2 - x_1 = \Delta x$, and so we can write speed as

$$u = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t}$$

since the speeds are constant as these are inertial frames.

Now let us call the speed of S', as seen in frame S, u, and call the speed of S'' as seen in S v.

$$v' = \frac{\Delta x'}{\Delta t'}$$

$$= \frac{\gamma(\Delta x + u\Delta t)}{\gamma\left(\Delta t + \frac{u\Delta x}{c^2}\right)}$$

Where we have used the Lorentz transformations for $\Delta x'$ and $\Delta t'$ noting the direction of v is opposite to u, hence the + instead of -. This is the speed of S'', in the frame S'.

Divide top and bottom by Δt:

$$v' = \frac{v+u}{1+\frac{vu}{c^2}}.$$

Which, substituting in the values for *u* and *v* gives $v' = 0.96c$.
This method of writing the speed as change in speed over time only works for constant velocities. It can be extended however to cover accelerating objects, by noting that as long as the acceleration is uniform $a = \frac{dv}{dt}$. N.B. we can transform accelerations between two

different inertial frames, but both of those frames cannot be accelerating, and we cannot transform into the accelerating frame (without using maths beyond this material). So we could ask how quickly a rocket was accelerating from the Earth frame compared to the frame of a space station flying away from the Earth at constant speed, but we could not find how quickly the space station was moving from the perspective of the accelerating space ship, as it is not an inertial frame.